"Special Continuous Distribution"

Rectangular Distribution (or Uniform Distribution)

Definition :-

A random variable X is said to have a continuous rectangular (uniform) distribution over an interval (a, b), i.e $(-\infty < a < b < \infty)$, if its pdf is given by

$$f(x; a, b) = \begin{cases} \frac{1}{b-a}, & \text{if } a < x < b \\ 0, & \text{otherwise.} \end{cases}$$

Remark: 1. 'a' and 'b', $(a<b)$ are the two parameters of the distribution. The distribution is called uniform distribution on (a,b) since it assumes a constant (uniform) value for all x in (a, b).

2. The distribution is also known as rectangular distribution, since the curve $y = f(x)$ describes a rectangle over the x-axis and ~~the~~ between the ordinates at $x = a$ and $x = b$.

3. A uniform (rectangular) variate X on the interval (a, b) is written as:

$$X \sim U[a, b] \text{ or } X \sim R[a, b]$$

4. **Distribution function:**

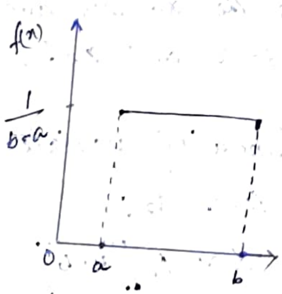
The cumulative distribution function $F(x)$ is given by

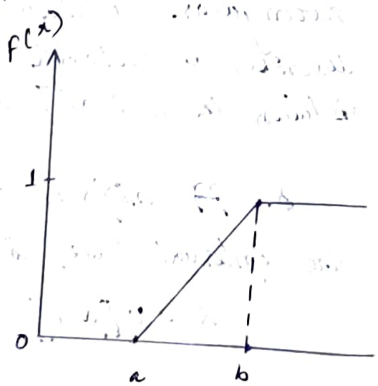
$$F(x) = \begin{cases} 0, & x \le a \\ \dfrac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b \end{cases}$$

Since $F(x)$ is not continuous at $x=a$ and $x=b$, it is not differentiable at these points.

Thus $\frac{d}{dx} F(x) = \frac{1}{b-a} = f(x) \neq 0$ exists everywhere except at the points $x=a$ and $x=b$.

5. The graphs of uniform pdf. $f(x)$ and the corresponding distribution function $F(x)$ are given below.





6. For a rectangular variate X in $(-a, a)$, the pdf is given by

$$f(x) = \begin{cases} \dfrac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise} \end{cases}$$

Moments of Rectangular distribution:

Let ~~Ut~~ $X \sim U[a,b]$

$$\mu'_r = \int_a^b x^r f(x)\,dx$$

$$= \int_a^b x^r \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a} \int_a^b x^r\,dx$$

$$= \frac{1}{b-a} \left(\frac{x^{r+1}}{r+1}\right)_a^b$$

$$= \frac{1}{b-a} \left[\frac{b^{r+1}}{r+1} - \frac{a^{r+1}}{r+1}\right]$$

$$= \frac{1}{b-a} \left(\frac{b^{r+1} - a^{r+1}}{r+1}\right)$$

In particular,

$$\mu'_1 = \text{Mean} = \frac{1}{b-a}\left(\frac{b^2-a^2}{2}\right) = \frac{b+a}{2}$$

$$\mu'_2 = \frac{1}{b-a}\left(\frac{b^3-a^3}{3}\right) = \frac{1}{3}(b^2+ab+a^2)$$

$\therefore$ Variance $= \mu_2 = \mu'_2 - (\mu'_1)^2$

$$= \cancel{\frac{b+a}{2}}\ \frac{1}{3}(b^2+ab+a^2) -$$

Moment Generating Function of Rectangular Distribution :

$$M_x(t) = E[e^{tx}]$$

$$= \int_a^b e^{tx} \cdot f(x)\, dx$$

$$= \int_a^b e^{tx} \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx}\, dx$$

$$= \frac{1}{b-a} \cdot \left(\frac{e^{tx}}{t}\right)_a^b$$

$$= \frac{1}{b-a}\left[\frac{e^{tb}}{t} - \frac{e^{ta}}{t}\right]$$

$$= \frac{e^{tb} - e^{ta}}{t(b-a)}, \quad t \neq 0. \quad //$$

Characteristic function of Uniform (Rectangular) Distribution :-

$$\phi_x(t) = E[e^{itx}]$$

$$= \int_a^b e^{itx} \cdot f(x)\, dx$$

$$= \int_a^b e^{itx} \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b e^{itx}\, dx$$

$$= \frac{1}{b-a} \left(\frac{e^{itx}}{it}\right)_a^b$$

$$= \frac{1}{b-a}\left[\frac{e^{itb}}{it} - \frac{e^{ita}}{it}\right]$$

$$= \frac{e^{itb} - e^{ita}}{it(b-a)}, \quad t \neq 0. \quad //$$

Mean deviation about mean :-

$$\eta = E|x - \text{Mean}|$$

$$= \int_a^b |x - \text{mean}|\, f(x)\, dx$$

$$= \int_a^b \left|x - \frac{a+b}{2}\right| \frac{1}{b-a}\, dx$$

Let $t = x - \frac{a+b}{2}$

$dt = dx$

when $x = a$, $t = \frac{a-b}{2}$

$x = b$, $t = \frac{b-a}{2}$

$$= \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} t \cdot \left(\frac{1}{b-a}\right) dt$$

$$= \frac{1}{b-a} \cdot \int_{\frac{a-b}{2}}^{\frac{b-a}{2}} t\, dt$$

$$= \frac{1}{b-a} \cdot 2 \int_0^{\frac{b-a}{2}} t\, dt$$

$$= \frac{1}{b-a} 2 \cdot \left(\frac{t^2}{2}\right)_0^{\frac{b-a}{2}}$$

$$= \frac{2}{b-a}\left(\frac{(b-a)^2/4}{2}\right)$$

$$= \frac{b-a}{4} \quad //$$

Q: If X is uniformly distributed with mean 1 and variance $\frac{4}{3}$, find $P(X<0)$:

Sol: Let $X \sim U[a,b]$,

$\therefore f(x) = \frac{1}{b-a}, \quad a<x<b.$

Given,

Mean $= \frac{a+b}{2} = 1$

$\Rightarrow a+b = 2 \longrightarrow (1)$

Variance$(X) = \frac{(b-a)^2}{12} = \frac{4}{3}$

$\Rightarrow (b-a)^2 = 16.$

$\Rightarrow b-a = \pm 4. \longrightarrow (2)$

Adding (1) and (2)

$a+b+b-a = 2+4$

$\Rightarrow 2b = 6$

$b = 3$

$\therefore a = -1 \quad (a<b).$

$\therefore f(x) = \frac{1}{3-(-1)}$

$= \frac{1}{4}, \quad -1 \le x \le 3.$

Now $P(X<0) = \int_{-1}^{0} f(x)\,dx$

$= \int_{-1}^{0} \frac{1}{4}\,dx$

$= \frac{1}{4}(0-(-1))$

$= \frac{1}{4} \quad //$