Discrete Uniform Distribution :-

A r.v X is said to have a discrete uniform distribution over the range $[1, n]$ if its pmf is expressed as follows:

$$P(X=x) = \begin{cases} \frac{1}{n}, & x = 1, 2, \cdots, n \\ 0, & \text{otherwise} \end{cases} \longrightarrow (1)$$

Here n is known as the parameter of the distribution and lies in the set of all positive integers. Then equation (1) is called discrete rectangular distribution.

Moments :-

$$E(X) = \frac{1}{n} \sum_{i=1}^{n} i$$

$$= \frac{1}{n}\left[1+2+3+\cdots+n\right]$$

$$= \frac{1}{n}\left[\frac{n(n+1)}{2}\right] = \frac{n+1}{2}.$$

$$E(X^r) = \frac{1}{n} \sum_{i=1}^{n} i^r$$

$$= \frac{1}{n}\left[1^r + 2^r + \cdots + n^r\right]$$

$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$V(x) = E(x^2) - \{E(x)\}^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{n+1}{2}\left[\frac{2n+1}{3} - \frac{n+1}{2}\right]$$

$$= \frac{n+1}{2}\left[\frac{4n+2-3n-3}{6}\right]$$

$$= \frac{n+1}{2}\left[\frac{n-1}{6}\right]$$

$$= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

Moment Generating function:

$$M_x(t) = E[e^{tx}]$$

$$= \frac{1}{n}\sum_{x=1}^{n} e^{tx}$$

$$= \frac{1}{n}\left[e^{t} + e^{2t} + e^{3t} + \cdots + e^{nt}\right]$$

$$= \frac{1}{n}\left[e^{t}(1 + e^{t} + e^{2t} + \cdots + e^{(n-1)t}\right]$$

$$= \frac{e^{t}(1-e^{nt})}{n(1-e^{t})}.$$

Bernoulli Distribution :-

A r.v X is said to have a Bernoulli distribution with parameter 'p' if its pmf is given by

$$P(X=x) = \begin{cases} p^{x}(1-p)^{1-x}, & \text{for } x = 0,1 \\ 0, & \text{otherwise.} \end{cases}$$

Moments of Bernoulli Distribution :-

$$E(X) = \sum_{x=0}^{1} x \cdot p(x)$$

$$= 0 \cdot p^{0}(1-p)^{1-0} + 1 \cdot p^{1}(1-p)^{1-1}$$

$$= 0 + p$$

$$= p.$$

$$E(X^2) = \sum_{x=0}^{1} x^2 \cdot p(X=x)$$

$$= 0^2 \cdot p^{0}(1-p)^{1-0} + 1^2 \cdot p^{1}(1-p)^{1-1}$$

$$= 0 + p = p$$

$$\therefore V(X) = E(X^2) - \{E(X)\}^2$$

$$= p - p^2$$

$$= p(1-p)$$

$$= pq \quad [\because 1-p = q]$$

Moment of Bernoulli variate is :-

$M_x(t) = E[e^{tx}]$

$= \sum_{x=0}^{1} e^{tx} \cdot p(x)$

$= \sum_{x=0}^{1} e^{tx} \cdot p^x (1-p)^{1-x}$

$= e^{t \cdot 0} p^0 (1-p)^{1-0} + e^{t \cdot 1} p (1-p)^{1-1}$

$= (1-p) + e^t (1) p$

$= q + pe^t$

Degenerate Random Variable :-

Some we may come across a variate X which is degenerate at c' say, so that $P(X=c)=1$ and 0 otherwise i.e the whole mass of the variable is concentrated at a single point c.

$P(X=c)=1, \quad Var(X)=0.$

Hence a degenerate r.v X is characterised by $Var(X)=0$.

Moment of degenerate r.v is

$M_X(t) = E[e^{tX}]$

$= e^{tc} P(X=c)$

$= e^{tc} \cdot 1 = e^{tc}$

Binomial distribution :-

Binomial distribution was discovered by James Bernoulli (1654-1705).

Let a random experiment be performed, each repetition being called a trial and let the occurrence of an event in a trial be called a success and its non-occurrence a failure.

Consider a set of n independent Bernoulli trials (n being finite) in which the probability 'p' of success in any trial is constant for each trial, then $q = 1-p$, is the probability of failure in any trial.

The probability of x successes and (n-x) failures in n independent trials, in specified order say SSFSFFFS... FSF (where S represents success and F represents failure) is given by the compound probability theorem by the expression:

$P(SSFSFFFS \ldots FSF) = P(S)\,P(S)\,P(F)\,P(S) \cdots P(F)\,P(S)$

$= p\,p\,q\,p \cdots q\,p$

$= p\,p \cdots p \cdot q\,q \cdots q$

$= p^x \cdot q^{n-x}$

But x success can occur only in ${}^nC_x$ ways and the probability for each of these ways is same viz $p^x q^{n-x}$.

Hence the probability of x successes in n trials in any order is given by the addition by the theorem of probability by the expression $\binom{n}{x} p^x q^{n-x}$.

Definition:-

A random variable X is said to follow Binomial distribution if it assumes only non negative values and its probability mass function is given by

$$P(X=x) = f(x) = \binom{n}{x} p^x q^{n-x}, \quad x = 0, 1, 2 \cdots n.$$

Binomial distribution is a discrete distribution as X can take only the integral values viz $0, 1, 2 \cdots n$. Any R.V which follows binomial distribution is a binomial variate.

Moments of Binomial distribution:-

$$\mu_1' = E(x) = \sum_{x=0}^{n} x \cdot \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n}{x} \binom{n-1}{x-1} p^x q^{n-x} \quad \left[\binom{n}{x} = \frac{n}{x}\binom{n-1}{x-1} = \frac{n}{x}\frac{n-1}{x-1}\binom{n-2}{x-2} \cdots\right]$$

$$= np \sum_{x=1}^{n} \binom{x-1}{x-1} p^{x-1} q^{n-x}$$

$$= np\,(q+p)^n \quad \left[\sum P(x) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (q+p)^n = 1\right]$$

$$= np$$

$$\mu_2' = E(x^2) = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \{x(x-1)+x\} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x} + \sum_{x=0}^{n} x\binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)\frac{n}{x}\frac{n-1}{x-1}\binom{n-2}{x-2} p^x q^{n-x} + np$$

$$= \sum_{x=2}^{n} n(n-1)p^2 \binom{n-2}{x-2} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 + np$$

$$\therefore \mu_2 = \mu_2' - (\mu_1')^2 = \mathrm{Var}(x)$$

$$= n(n-1)p^2 + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p)$$

$$= npq$$